A discrepancy principle for equations with monotone continuous operators

N. S. Hoang†*and A. G. Ramm†‡

†Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA

Abstract

A discrepancy principle for solving nonlinear equations with monotone operators given noisy data is formulated. The existence and uniqueness of the corresponding regularization parameter $a(\delta)$ is proved. Convergence of the solution obtained by the discrepancy principle is justified. The results are obtained under natural assumptions on the nonlinear operator.

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1 Introduction

Consider the equation:

$$F(u) = f, \tag{1}$$

where F is a monotone operator in a real Hilbert space H. Monotonicity is understood in the following sense:

$$\langle F(u) - F(v), u - v \rangle \geq 0, \quad \forall u, v \in H. \tag{2}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in H. Assume that F is continuous.

Equations with monotone operators are important in many applications and were studied extensively, see, for example, [1]–[3], [9], [10], [12], and references therein. There are many technical and physical problems leading to equations with such operators in the cases when dissipation of energy occurs. For example, in [5] and [4], Chapter 3, pp.156-189, a wide class of nonlinear dissipative systems is studied, and the basic equations of such systems can be reduced to equation (1) with monotone operators. Many examples of equations with monotone operators can be found in [2] and in references mentioned above. In [6] and [7] it is proved that any solvable linear operator equation with a closed densely defined operator in a Hilbert space H can be reduced to an equation with a monotone operator and solved by a convergent iterative process.

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‡Corresponding author. Email: ramm@math.ksu.edu

*Email: nguyenhs@math.ksu.edu

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In this paper, apparently for the first time, a discrepancy principle for solving equation (3) with noisy data (see Section 2) is proved under natural assumptions. No smallness assumptions on the nonlinearity, no global restrictions on its growth, or other special properties of the nonlinearity, except the monotonicity and continuity, are imposed. No source-type assumptions are used. Our result is widely applicable. It is well known that without extra assumptions, usually source-type assumption concerning the right-hand side, or some equivalent assumption concerning the smoothness of the solution, one cannot get a rate of convergence even for linear ill-posed equations (see, for example, [9]). On the other hand, such assumptions are usually not algorithmically verifiable and often they do not hold. By this reason we do not make such assumptions and do not give estimates of the rate of convergence.

In [11] a stationary equation $F(u) = f$ with a nonlinear monotone operator F was studied. The assumptions A1-A3 on p.197 in [11] are more restrictive than ours, and the Rule R2 on p.199, formula (4.1) in [11], for the choice of the regularization parameter is more difficult to use computationally: one has to solve nonlinear equation (4.1) in [11] for the regularization parameter. Moreover, to use this equation one has to invert an ill-conditioned linear operator $A + aI$ for small values of a. Assumption A1 in [11] is not verifiable, because the solution $x^\dagger$ is not known. Assumption A3 in [11] requires F to be constant in a ball $B_r(x^\dagger)$ if $F'(x^\dagger) = 0$. Our discrepancy principle does not require these assumptions, and, in contrast to equation (4.1) in [11], it does not require inversion of ill-conditioned linear operators.

The novel results in our paper include Theorem 5 in Section 3 and Theorem 7 in Section 4. In Theorem 5 a new discrepancy principle is proposed and justified assuming only the monotonicity and continuity of F. Implementing the discrepancy principle in Theorem 5 requires solving equation (3) and then solving nonlinear equation (15) for the regularization parameter $a(\delta)$. Theorem 7 allows one to solve equations (3) and (15) approximately. Thus, when δ is not too small one can save a large amount of computations in solving equations (3) and (15) by applying Theorem 7 and using our new stopping rule. Our results allow one to solve numerically stably equation (1) if F is locally Lipschitz and monotone. Based on Theorem 7, an algorithm for stable solution of equation (1) is formulated for locally Lipschitz monotone operators.

2   Auxiliary results

Let us consider the following equation

$$F(V_{\delta,a}) + aV_{\delta,a} - f_\delta = 0, \qquad a > 0, \tag{3}$$

where $a = const$. It is known (see, e.g., [9, p.111]) that equation (3) with monotone continuous operator F has a unique solution for any $f_\delta \in H$.

Throughout the paper we assume that F is a monotone continuous operator and the inner product in H is denoted $\langle u, v \rangle$. Below the word decreasing means strictly decreasing and increasing means strictly increasing.

Recall the following result from [9, p.112]:

Lemma 1 *Assume that equation (1) is solvable, y is its minimal-norm solution, assumption (2) holds, and F is continuous. Then*

$$\lim_{a \to 0} \|V_a - y\| = 0, \tag{4}$$

where V_a solves equation (3) with $\delta = 0$.

Lemma 2 *Assume $\|F(0) - f_\delta\| > 0$. Let $a > 0$, and F be monotone. Denote*

$$\psi(a) := \|V_{\delta,a}\|, \qquad \phi(a) := a\psi(a) = \|F(V_{\delta,a}) - f_\delta\|,$$

where $V_{\delta,a}$ solves (3). Then $\psi(a)$ is decreasing, and $\phi(a)$ is increasing.

Proof. Since $\|F(0) - f_\delta\| > 0$, one has $\psi(a) \neq 0$, $\forall a \geq 0$. Indeed, if $\psi(a)\big|_{a=\tau} = 0$, then $V_{\delta,a} = 0$, and equation (3) implies $\|F(0) - f_\delta\| = 0$, which is a contradiction. Note that $\phi(a) = a\|V_{\delta,a}\|$. One has

$$
\begin{aligned}
0 &\leq \langle F(V_{\delta,a}) - F(V_{\delta,b}), V_{\delta,a} - V_{\delta,b} \rangle \\
&= \langle -aV_{\delta,a} + bV_{\delta,b}, V_{\delta,a} - V_{\delta,b} \rangle \\
&= (a+b)\langle V_{\delta,a}, V_{\delta,b} \rangle - a\|V_{\delta,a}\|^2 - b\|V_{\delta,b}\|^2.
\end{aligned}
\tag{5}
$$

Thus,

$$
\begin{aligned}
0 &\leq (a+b)\langle V_{\delta,a}, V_{\delta,b} \rangle - a\|V_{\delta,a}\|^2 - b\|V_{\delta,b}\|^2 \\
&\leq (a+b)\|V_{\delta,a}\|\|V_{\delta,b}\| - a\|V_{\delta,a}\|^2 - b\|V_{\delta,b}\|^2 \\
&= (a\|V_{\delta,a}\| - b\|V_{\delta,b}\|)(\|V_{\delta,b}\| - \|V_{\delta,a}\|) \\
&= (\phi(a) - \phi(b))(\psi(a) - \psi(b)).
\end{aligned}
\tag{6}
$$

If $\psi(b) > \psi(a)$ then (6) implies $\phi(a) \geq \phi(b)$, so

$$a\psi(a) \geq b\psi(b) > b\psi(a).$$

Therefore, if $\psi(b) > \psi(a)$ then $b < a$.

Similarly, if $\psi(b) < \psi(a)$ then $\phi(a) \leq \phi(b)$. This implies $b > a$.

Suppose $\psi(a) = \psi(b)$, i.e., $\|V_{\delta,a}\| = \|V_{\delta,b}\|$. From (5) one has

$$\|V_{\delta,a}\|^2 \leq \langle V_{\delta,a}, V_{\delta,b} \rangle \leq \|V_{\delta,a}\|\|V_{\delta,b}\| = \|V_{\delta,a}\|^2.$$

This implies $V_{\delta,a} = V_{\delta,b}$, and then equation (3) implies $a = b$.

Therefore ϕ is increasing and ψ is decreasing. $\qquad \square$

Lemma 3 *If F is monotone and continuous, then $\|V_{\delta,a}\| = O(\frac{1}{a})$ as $a \to \infty$, and*

$$\lim_{a \to \infty} \|F(V_{\delta,a}) - f_\delta\| = \|F(0) - f_\delta\|. \tag{7}$$

Proof. Rewrite (3) as

$$F(V_{\delta,a}) - F(0) + aV_{\delta,a} + F(0) - f_\delta = 0.$$

Multiply this equation by $V_{\delta,a}$, use the monotonicity of F and get:

$$a\|V_{\delta,a}\|^2 \leq \langle aV_{\delta,a} + F(V_{\delta,a}) - F(0), V_{\delta,a}\rangle = \langle f_\delta - F(0), V_{\delta,a}\rangle \leq \|f_\delta - F(0)\|\|V_{\delta,a}\|.$$

Therefore, $\|V_{\delta,a}\| = O(\frac{1}{a})$. This and the continuity of F imply (7). $\qquad\square$

Remark 1 If $\|F(0) - f_\delta\| > C\delta^\gamma, 0 < \gamma \leq 1$ then relation (7) implies

$$\|F(V_{\delta,a}) - f_\delta\| \geq C\delta^\gamma, \qquad 0 < \gamma \leq 1, \tag{8}$$

for sufficiently large $a > 0$.

Lemma 4 *Let $C > 0$ and $\gamma \in (0,1]$ be constants such that $C\delta^\gamma > \delta$. Suppose that $\|F(0) - f_\delta\| > C\delta^\gamma$. Then, there exists a unique $a(\delta) > 0$ such that $\|F(V_{\delta,a(\delta)}) - f_\delta\| = C\delta^\gamma$.*

Proof. We have $F(y) = f$, and

$$\begin{aligned}
0 =& \langle F(V_{\delta,a}) + aV_{\delta,a} - f_\delta, F(V_{\delta,a}) - f_\delta\rangle\\
=& \|F(V_{\delta,a}) - f_\delta\|^2 + a\langle V_{\delta,a} - y, F(V_{\delta,a}) - f_\delta\rangle + a\langle y, F(V_{\delta,a}) - f_\delta\rangle\\
=& \|F(V_{\delta,a}) - f_\delta\|^2 + a\langle V_{\delta,a} - y, F(V_{\delta,a}) - F(y)\rangle + a\langle V_{\delta,a} - y, f - f_\delta\rangle\\
&+ a\langle y, F(V_{\delta,a}) - f_\delta\rangle\\
\geq& \|F(V_{\delta,a}) - f_\delta\|^2 + a\langle V_{\delta,a} - y, f - f_\delta\rangle + a\langle y, F(V_{\delta,a}) - f_\delta\rangle.
\end{aligned}$$

Here the monotonicity of F was used. Therefore

$$\begin{aligned}
\|F(V_{\delta,a}) - f_\delta\|^2 &\leq -a\langle V_{\delta,a} - y, f - f_\delta\rangle - a\langle y, F(V_{\delta,a}) - f_\delta\rangle\\
&\leq a\|V_{\delta,a} - y\|\|f - f_\delta\| + a\|y\|\|F(V_{\delta,a}) - f_\delta\| \qquad (9)\\
&\leq a\delta\|V_{\delta,a} - y\| + a\|y\|\|F(V_{\delta,a}) - f_\delta\|.
\end{aligned}$$

Also,

$$\begin{aligned}
0 &= \langle F(V_{\delta,a}) - F(y) + aV_{\delta,a} + f - f_\delta, V_{\delta,a} - y\rangle\\
&= \langle F(V_{\delta,a}) - F(y), V_{\delta,a} - y\rangle + a\|V_{\delta,a} - y\|^2 + a\langle y, V_{\delta,a} - y\rangle + \langle f - f_\delta, V_{\delta,a} - y\rangle\\
&\geq a\|V_{\delta,a} - y\|^2 + a\langle y, V_{\delta,a} - y\rangle + \langle f - f_\delta, V_{\delta,a} - y\rangle,
\end{aligned}$$

where the monotonicity of F was used again. Therefore,

$$a\|V_{\delta,a} - y\|^2 \leq a\|y\|\|V_{\delta,a} - y\| + \delta\|V_{\delta,a} - y\|.$$

This implies

$$a\|V_{\delta,a} - y\| \leq a\|y\| + \delta. \tag{10}$$

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From (9), (10), and an elementary inequality $ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$, $\forall \epsilon > 0$, one gets:

$$\|F(V_{\delta,a}) - f_\delta\|^2 \leq \delta^2 + a\|y\|\delta + a\|y\|\|F(V_{\delta,a}) - f_\delta\|$$
$$\leq \delta^2 + a\|y\|\delta + \epsilon\|F(V_{\delta,a}) - f_\delta\|^2 + \frac{1}{4\epsilon}a^2\|y\|^2, \qquad (11)$$

where $\epsilon > 0$ is arbitrary small, fixed, independent of a, and can be chosen arbitrary small. Let $a \searrow 0$. Then (11) implies $\lim_{a \to 0}(1 - \epsilon)\|F(V_{\delta,a}) - f_\delta\|^2 \leq \delta^2 < (C\delta^\gamma)^2$. Thus,

$$\lim_{a \to 0} \|F(V_{\delta,a}) - f_\delta\| < C\delta^\gamma, \qquad C > 0, \quad 0 < \gamma \leq 1.$$

This, the continuity of F, the continuity of $V_{\delta,a}$ with respect to $a \in [0, \infty)$, and inequality (8), imply that equation $\|F(V_{\delta,a}) - f_\delta\| = C\delta^\gamma$ must have a solution $a(\delta) > 0$. $\qquad \square$

Remark 2 Let $V_a := V_{\delta,a}|_{\delta=0}$, so $F(V_a) + aV - f = 0$. Let y be the minimal-norm solution to equation (1). We claim that

$$\|V_{\delta,a} - V_a\| \leq \frac{\delta}{a}. \qquad (12)$$

Indeed, from (3) one gets

$$F(V_{\delta,a}) - F(V_a) + a(V_{\delta,a} - V_a) = f - f_\delta.$$

Multiply this equality by $(V_{\delta,a} - V_a)$ and use (2) to obtain

$$\delta\|V_{\delta,a} - V_a\| \geq \langle f - f_\delta, V_{\delta,a} - V_a \rangle$$
$$= \langle F(V_{\delta,a}) - F(V_a) + a(V_{\delta,a} - V_a), V_{\delta,a} - V_a \rangle$$
$$\geq a\|V_{\delta,a} - V_a\|^2.$$

This implies (12).

Let us derive a uniform with respect to a bound on $\|V_a\|$. From the equation

$$F(V_a) + aV_a - F(y) = 0,$$

and the monotonicity of F one gets

$$0 = \langle F(V_a) + aV_a - F(y), V_a - y \rangle \geq a\langle V_a, V_a - y \rangle.$$

This implies the desired bound:

$$\|V_a\| \leq \|y\|, \qquad \forall a > 0. \qquad (13)$$

Similar arguments one can find in [9, p. 113].

From (12) and (13), one gets the following estimate:

$$\|V_{\delta,a}\| \leq \|V_a\| + \frac{\delta}{a} \leq \|y\| + \frac{\delta}{a}. \qquad (14)$$

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3 A discrepancy principle

Our standing assumptions are the monotonicity and continuity of F and the solvability of equation (1). They are not repeated below. We assume without loss of generality that $\delta \in (0,1)$.

Theorem 5 *Let $\gamma \in (0,1]$ and $C > 0$ be some constants such that $C\delta^\gamma > \delta$. Assume that $\|F(0) - f_\delta\| > C\delta^\gamma$. Let y be its minimal-norm solution. Then there exists a unique $a(\delta) > 0$ such that*

$$\|F(V_{\delta,a(\delta)}) - f_\delta\| = C\delta^\gamma, \tag{15}$$

where $V_{\delta,a(\delta)}$ solves (3) with $a = a(\delta)$.
  If $0 < \gamma < 1$ then

$$\lim_{\delta \to 0} \|V_{\delta,a(\delta)} - y\| = 0. \tag{16}$$

Proof. The existence and uniqueness of $a(\delta)$ follow from Lemma 4. Let us show that

$$\lim_{\delta \to 0} a(\delta) = 0. \tag{17}$$

The triangle inequality, inequality (12) and equality (15) imply

$$a(\delta)\|V_{a(\delta)}\| \le a(\delta)\big(\|V_{\delta,a(\delta)} - V_{a(\delta)}\| + \|V_{\delta,a(\delta)}\|\big)$$
$$\le \delta + a(\delta)\|V_{\delta,a(\delta)}\| = \delta + C\delta^\gamma. \tag{18}$$

From inequality (18), one gets

$$\lim_{\delta \to 0} a(\delta)\|V_{a(\delta)}\| = 0. \tag{19}$$

It follows from Lemma 2 with $f_\delta = f$, i.e., $\delta = 0$, that the function $\phi_0(a) := a\|V_a\|$ is nonnegative and strictly increasing on $(0,\infty)$. This and relation (19) imply:

$$\lim_{\delta \to 0} a(\delta) = 0. \tag{20}$$

From (15) and (14), one gets

$$C\delta^\gamma = a\|V_{\delta,a}\| \le a(\delta)\|y\| + \delta. \tag{21}$$

Thus, one gets:

$$C\delta^\gamma - \delta \le a(\delta)\|y\|. \tag{22}$$

If $\gamma < 1$ then $C - \delta^{1-\gamma} > 0$ for sufficiently small δ. This implies:

$$0 \le \lim_{\delta \to 0} \frac{\delta}{a(\delta)} \le \lim_{\delta \to 0} \frac{\delta^{1-\gamma}\|y\|}{C - \delta^{1-\gamma}} = 0. \tag{23}$$

By the triangle inequality and inequality (12), one has

$$\|V_{\delta,a(\delta)} - y\| \le \|V_{a(\delta)} - y\| + \|V_{a(\delta)} - V_{\delta,a(\delta)}\| \le \|V_{a(\delta)} - y\| + \frac{\delta}{a(\delta)}. \tag{24}$$

Relation (16) follows from (23), (24) and Lemma 1. $\qquad\square$

Instead of using (3), one may use the following equation:

$$F(V_{\delta,a}) + a(V_{\delta,a} - \bar{u}) - f_\delta = 0, \qquad a > 0, \tag{25}$$

where $\bar{u}$ is an element of H. Denote $F_1(u) := F(u + \bar{u})$. Then F_1 is monotone and continuous. Equation (3) can be written as:

$$F_1(U_{\delta,a}) + aU_{\delta,a} - f_\delta = 0, \qquad U_{\delta,a} := V_{\delta,a} - \bar{u}, \quad a > 0. \tag{26}$$

By applying Theorem 5 with $F = F_1$ one gets the following result:

Corollary 6 *Let* $\gamma \in (0,1]$ *and* $C > 0$ *be some constants such that* $C\delta^\gamma > \delta$. *Let* $\bar{u} \in H$ *and* z *be the solution to* (1) *with minimal distance to* $\bar{u}$. *Assume that* $\|F(\bar{u}) - f_\delta\| > C\delta^\gamma$. *Then there exists a unique* $a(\delta) > 0$ *such that*

$$\|F(\tilde{V}_{\delta,a(\delta)}) - f_\delta\| = C\delta^\gamma, \tag{27}$$

where $\tilde{V}_{\delta,a(\delta)}$ *solves the following equation:*

$$F(\tilde{V}_{\delta,a}) + a(\delta)(\tilde{V}_{\delta,a} - \bar{u}) - f_\delta = 0.$$

If $\gamma \in (0,1)$ *then this* $a(\delta)$ *satisfies*

$$\lim_{\delta \to 0} \|\tilde{V}_{\delta,a(\delta)} - z\| = 0. \tag{28}$$

Remark 3 It is an open problem to choose γ and C optimal in some sense.

Remark 4 Theorem 5 and Theorem 7 do not hold, in general, for $\gamma = 1$. Indeed, let $Fu = \langle u, p \rangle p$, $\|p = 1\|$, $p \perp \mathcal{N}(F) := \{u \in H : Fu = 0\}$, $f = p$, $f_\delta = p + q\delta$, where $\langle p, q \rangle = 0$, $\|q\| = 1$, $Fq = 0$, $\|q\delta\| = \delta$. One has $Fy = p$, where $y = p$, is the minimal-norm solution to the equation $Fu = p$. Equation $Fu + au = p + q\delta$, has the unique solution $V_{\delta,a} = q\delta/a + p/(1 + a)$. Equation (15) is $C\delta = \|q\delta + (ap)/(1 + a)\|$. This equation yields $a = a(\delta) = c\delta/(1 - c\delta)$, where $c := (C^2 - 1)^{1/2}$, and we assume $c\delta < 1$. Thus, $\lim_{\delta \to 0} V_{\delta,a(\delta)} = p + c^{-1}q := v$, and $Fv = p$. Therefore $v = \lim_{\delta \to 0} V_{\delta,a(\delta)}$ is not p, i.e., is not the minimal-norm solution to the equation $Fu = p$. Similar arguments one can find in [8, p. 29].

4 Applications

In this section we discuss methods for solving equations (3) and (1) using the new discrepancy principle, i.e., Theorem 5. Implementing this principle, i.e., solving equation (15), requires solving equation (3). *If* F *is linear*, then equation (3) has the form:

$$(F + aI)u = f_\delta. \tag{29}$$

Since $F \geq 0$ the operator $F + aI$ is boundedly invertible, $\|(F + aI)^{-1}\| \leq \frac{1}{a}$, and equation (29) is well-posed if $a > 0$ is not too small. There are many methods for solving efficiently

well-posed linear equations with positive-definite operators. For this reason we *mainly discuss some methods for stable solution of equation* (1) *with nonlinear operators.* In this section a method is developed for a stable solution of equation (1) with locally Lipschitz monotone operator F, so we assume that

$$\|F(u) - F(v)\| \leq L\|u - v\|, \quad u, v \in B(u_0, R) := \{u : \|u - u_0\| \leq R\}, \quad L = L(R). \quad (30)$$

Here $u_0 \in H$ is an arbitrary fixed element. Consider the operator

$$G(u) := u - \lambda[F(u) + au - f_\delta], \quad \lambda > 0.$$

We claim that G is a contraction mapping in H provided that λ is sufficiently small. Let $F_1 := F + aI$. Then (30) implies $\|F_1(u) - F_1(v)\| \leq (a+L)\|u-v\|$. Using the monotonicity of F, one gets

$$\begin{aligned}
\|G(u) - G(v)\|^2 &= \|(u - v) - \lambda(F_1(u) - F_1(v))\|^2 \\
&= \|u - v\|^2 - 2\lambda\langle u - v, F_1(u) - F_1(v)\rangle + \lambda^2\|F_1(u) - F_1(v)\|^2 \quad (31) \\
&\leq \|u - v\|^2[1 - 2\lambda a + \lambda^2(a + L)^2].
\end{aligned}$$

This implies that G is a contraction mapping if

$$0 < \lambda < \frac{2a}{(a + L)^2}.$$

For these λ the solution $V_{\delta,a}$ of equation (3) can be found by the following iterative process:

$$u_{n+1} = u_n - \lambda[F(u_n) + au_n - f_\delta], \quad u_0 := u_0. \quad (32)$$

After finding $V_{\delta,a}$, one finds $a(\delta)$ from the discrepancy principle (15), i.e., by solving the nonlinear equation:

$$\phi(a(\delta)) := \|F(V_{\delta,a(\delta)}) - f_\delta\| = C\delta^\gamma. \quad (33)$$

There are many methods for solving this equation. For example, one can use the bisection method or the golden section method. If $a(\delta)$ is found, one solves equation (3) with $a = a(\delta)$ for $V_{\delta,a(\delta)}$ and takes its solution as an approximate solution to (1).

Although the sequence u_n, defined by (32), converges to the solution of equation (3) at the rate of a geometrical series with a denominator $q \in (0, 1)$, it is very time consuming to try to solve equation (3) with high accuracy if q is close to 1. Theorem 7 (see below) allows one to stop iterations (32) at the first value of n which satisfies the following condition:

$$\|F(u_n) + au_n - f_\delta\| \leq \theta\delta, \quad \theta > 0, \quad (34)$$

where θ is a fixed constant. This saves the time of computation.

Theorem 7 *Let δ, F, f_δ, and y be as in Theorem 5 and $0 < \gamma < 1$. Assume that $v_\delta \in H$ and $\alpha(\delta) > 0$ satisfy the following conditions:*

$$\|F(v_\delta) + \alpha(\delta)v_\delta - f_\delta\| \leq \theta\delta, \quad \theta > 0, \quad (35)$$

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and

$$C_1\delta^\gamma \leq \|F(v_\delta) - f_\delta\| \leq C_2\delta^\gamma, \qquad 0 < C_1 < C_2. \tag{36}$$

Then one has:

$$\lim_{\delta \to 0} \|v_\delta - y\| = 0. \tag{37}$$

Proof. Let u and v be arbitrary elements in H. By the monotonicity of F one gets

$$\begin{aligned} a\|u - v\|^2 &\leq \langle u - v, F(u) - F(v) + au - av \rangle \\ &\leq \|u - v\|\|F(u) - F(v) + au - av\|, \qquad \forall a > 0. \end{aligned} \tag{38}$$

This implies

$$a\|u - v\| \leq \|F(u) - F(v) + au - av\|, \qquad \forall v, u \in H, \quad \forall a > 0. \tag{39}$$

Using inequality (39) with $v = v_\delta$ and $u = V_{\delta,\alpha(\delta)}$, equation (3) with $a = \alpha(\delta)$, and inequality (35), one gets

$$\begin{aligned} \alpha(\delta)\|v_\delta - V_{\delta,\alpha(\delta)}\| &\leq \|F(v_\delta) - F(V_{\delta,\alpha(\delta)}) + \alpha(\delta)v_\delta - \alpha(\delta)V_{\delta,\alpha(\delta)}\| \\ &= \|F(v_\delta) + \alpha(\delta)v_\delta - f_\delta\| \leq \theta\delta. \end{aligned} \tag{40}$$

Therefore,

$$\|v_\delta - V_{\delta,\alpha(\delta)}\| \leq \frac{\theta\delta}{\alpha(\delta)}. \tag{41}$$

Using (14) and (41), one gets:

$$\alpha(\delta)\|v_\delta\| \leq \alpha(\delta)\|V_{\delta,\alpha(\delta)}\| + \alpha(\delta)\|v_\delta - V_{\delta,\alpha(\delta)}\| \leq \theta\delta + \alpha(\delta)\|y\| + \delta. \tag{42}$$

From the triangle inequality and inequalities (35) and (36) one obtains:

$$\alpha(\delta)\|v_\delta\| \geq \|F(v_\delta) - f_\delta\| - \|F(v_\delta) + \alpha(\delta)v_\delta - f_\delta\| \geq C_1\delta^\gamma - \theta\delta. \tag{43}$$

Inequalities (42) and (43) imply

$$C_1\delta^\gamma - \theta\delta \leq \theta\delta + \alpha(\delta)\|y\| + \delta. \tag{44}$$

This inequality and the fact that $C_1 - \delta^{1-\gamma} - 2\theta\delta^{1-\gamma} > 0$ for sufficiently small δ and $0 < \gamma < 1$ imply

$$\frac{\delta}{\alpha(\delta)} \leq \frac{\delta^{1-\gamma}\|y\|}{C_1 - \delta^{1-\gamma} - 2\theta\delta^{1-\gamma}}, \qquad 0 < \delta \ll 1. \tag{45}$$

Thus, one obtains

$$\lim_{\delta \to 0} \frac{\delta}{\alpha(\delta)} = 0. \tag{46}$$

From the triangle inequality and inequalities (35), (36) and (41), one gets

$$\begin{aligned} \alpha(\delta)\|V_{\delta,\alpha(\delta)}\| &\leq \|F(v_\delta) - f_\delta\| + \|F(v_\delta) + \alpha(\delta)v_\delta - f_\delta\| + \alpha(\delta)\|v_\delta - V_{\delta,\alpha(\delta)}\| \\ &\leq C_2\delta^\gamma + \theta\delta + \theta\delta. \end{aligned} \tag{47}$$

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This inequality implies
$$\lim_{\delta \to 0} \alpha(\delta)\|V_{\delta,\alpha(\delta)}\| = 0. \tag{48}$$

The triangle inequality and inequality (12) imply
$$\begin{aligned}
\alpha\|V_\alpha\| &\le \alpha\big(\|V_{\delta,\alpha} - V_\alpha\| + \|V_{\delta,\alpha}\|\big) \\
&\le \delta + \alpha\|V_{\delta,\alpha}\|.
\end{aligned} \tag{49}$$

From formulas (49) and (48), one gets
$$\lim_{\delta \to 0} \alpha(\delta)\|V_{\alpha(\delta)}\| = 0. \tag{50}$$

It follows from Lemma 2 with $f_\delta = f$, i.e., $\delta = 0$, that the function $\phi_0(a) := a\|V_a\|$ is nonnegative and strictly increasing on $(0, \infty)$. This and relation (50) imply
$$\lim_{\delta \to 0} \alpha(\delta) = 0. \tag{51}$$

From the triangle inequality and inequalities (41) and (12) one obtains
$$\begin{aligned}
\|v_\delta - y\| &\le \|v_\delta - V_{\delta,\alpha(\delta)}\| + \|V_{\delta,\alpha(\delta)} - V_{\alpha(\delta)}\| + \|V_{\alpha(\delta)} - y\| \\
&\le \frac{\theta\delta}{\alpha(\delta)} + \frac{\delta}{\alpha(\delta)} + \|V_{\alpha(\delta)} - y\|,
\end{aligned} \tag{52}$$

where $V_{\alpha(\delta)}$ solves equation (3) with $a = \alpha(\delta)$ and $f_\delta = f$.

The conclusion (37) follows from inequalities (46), (51), (52) and Lemma 1. Theorem 7 is proved. $\square$

Remark 5 Inequalities (35) and (36) are used as stopping rules for finding approximations:
$$\alpha(\delta) \approx a(\delta), \quad \text{and} \quad v(\delta) \approx V_{\delta,a(\delta)}.$$

Remark 6 By the monotonicity of F one gets
$$\begin{aligned}
\|F(u) - F(v)\|^2 &\le \langle F(u) - F(v), F(u) - F(v) + a(u - v)\rangle \\
&\le \|F(u) - F(v)\|\|F(u) - F(v) + a(u - v)\|, \quad \forall u, v \in H, \quad \forall a > 0.
\end{aligned}$$

This implies
$$\|F(u) - F(v)\| \le \|F(u) - F(v) + a(u - v)\|, \qquad \forall u, v \in H, \quad a > 0. \tag{53}$$

Fix $\delta > 0$ and $\theta > 0$. Let C be as in Theorem 5. Choose C_1 and C_2 such that
$$C_1\delta^\gamma + \theta\delta < C\delta^\gamma < C_2\delta^\gamma - \theta\delta. \tag{54}$$

Suppose α_i and v_i, $i = 1, 2$, satisfy condition (35) and
$$\|F(v_1) - f_\delta\| < C_1\delta^\gamma, \qquad C_2\delta^\gamma < \|F(v_2) - f_\delta\|. \tag{55}$$

Let us show that
$$\alpha_{low} := \alpha_1 < a(\delta) < \alpha_2 := \alpha_{up}, \tag{56}$$
where $a(\delta)$ satisfies conditions of Theorem 5. Using inequality (53) for v_i and V_{δ,α_i}, $i = 1, 2$, and inequality (35), one gets
$$\begin{aligned}
\|F(v_i) - F(V_{\delta,\alpha_i})\| &\le \|F(v_i) - F(V_{\delta,\alpha_i}) + \alpha_i v_i - \alpha_i V_{\delta,\alpha_i}\| \\
&\le \|F(v_i) + \alpha_i v_i - f_\delta\| \le \theta\delta.
\end{aligned} \tag{57}$$

From inequalities (55), (57) and the triangle inequality, one derives:
$$\|F(V_{\delta,\alpha_1}) - f_\delta\| < C_1\delta^\gamma + \theta\delta \quad \text{and} \quad C_2\delta^\gamma - \theta\delta < \|F(V_{\delta,\alpha_2}) - f_\delta\|. \tag{58}$$

Recall that $\|F(V_{\delta,a(\delta)}) - f_\delta\| = C\delta^\gamma$. Inequality (56) is obtained from inequalities (54), (58) and the fact that the function $\phi(\alpha) = \|F(V_{\delta,\alpha}) - f_\delta\|$ is strictly increasing (see Lemma 2).

Let $f_\delta, F, C, \theta, \gamma$, and δ be as in Theorem 5 and 7, and C_1 and C_2 satisfy inequality (54). Let us formulate an algorithm (see **Algorithm 1** below) for finding $\alpha(\delta) \approx a(\delta)$ and $v(\delta) \approx V_{\delta,a(\delta)}$, using the bisection method and assuming that F is a locally Lipschitz monotone operator and α_{low} and α_{up} are known. By Theorem 7, $v(\delta)$ can be considered as a stable solution to equation (1).

Algorithm 1: *Finding $\alpha(\delta) \approx a(\delta)$ and $v_\delta \approx V_{\delta,a(\delta)}$ given α_{low} and α_{up}.*

1. Let $a := \frac{\alpha_{up} + \alpha_{low}}{2}$ and u_0 be an initial guess for $V_{\delta,a}$. Compute u_n by formula (32) and stop at n_{stop}, where n_{stop} is the smallest $n > 0$ for which condition (35) is satisfied. Then go to step 2.

2. If $C_2\delta^\gamma < \|F(u_{n_{stop}}) - f_\delta\|$, then set $\alpha_{up} := a$ and go to step 4. Otherwise, go to step 3.

3. If $C_1\delta^\gamma \le \|F(u_{n_{stop}}) - f_\delta\|$, then stop the process and take $v(\delta) := u_{n_{stop}}$ as a solution to (1). If $\|F(u_{n_{stop}}) - f_\delta\| < C_1\delta^\gamma$, then set $\alpha_{low} := a$ and go to step 4.

4. Check if $\|a - \alpha_{low}\|$ is less than a desirable small value $\epsilon > 0$. If it is, then take $v(\delta) := u_{n_{stop}}$ as a solution to (1). If is is not, then go back to step 1.

Let us formulate algorithms for finding α_{up} and α_{low}.
Algorithm 2: *Finding α_{up}.*

1. Let $a = \alpha$ be an initial guess for $\alpha(\delta)$ and u_0 be an initial guess for v_δ. Compute u_n by formula (32) with a and stop at n_{stop}, the smallest $n > 0$ for which condition (35) is satisfied. Then go to step 2.

2. If condition (36) holds for $v_\delta := u_{n_{stop}}$, then stop the process and take $u_{n_{stop}}$ as a solution to (1). Otherwise, go to step 3.

3. If $C_2\delta^\gamma < \|F(u_{n_{stop}}) - f_\delta\|$, then set $\alpha_{up} := a$. Otherwise, set $\alpha := 2a$ and go back step 1.

Algorithm 3: *Finding α_{low}.*

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1. Let $a = \alpha$ be an initial guess for $\alpha(\delta)$ and u_0 be an initial guess for v_δ. Compute u_n by formula (32) with a and stop at n_{stop}, the smallest $n > 0$ for which condition (35) is satisfied. Then go to step 2.

2. If condition (36) holds for $v_\delta := u_{n_{stop}}$, then stop the process and take $u_{n_{stop}}$ as a solution to (1). Otherwise, go to step 3.

3. If $\|F(u_{n_{stop}}) - f_\delta\| < C_1 \delta^\gamma$, then set $\alpha_{low} := a$. Otherwise, set $\alpha := \frac{a}{2}$ and go back step 1.

In practice these algorithms are often implemented at the same time to avoid repetition calculations.

Remark 7 The sequence $(\|u_n - V_{\delta,a(\delta)}\|)_{n=0}^\infty$, where u_n is computed by formula (32) and $V_{\delta,a(\delta)}$ is the solution to (3) with $a = a(\delta)$, is decreasing. Thus, the sequence u_n will stay inside a ball $B(0, R)$ assuming that $R > 0$ is chosen sufficiently large, so that $y, u_0 \in B(0, R)$.

Remark 8 Theorem 7 and the above algorithms are not only useful for solving nonlinear equations with monotone operators but also for solving linear equations with monotone operators. If one uses iterative methods to solve equation (29) then, by using Theorem 7, one can stop iterations whenever inequality (35) holds. By using stopping rule (35) one saves time of computations compared to solving (29) exactly. If F is a positive matrix then one can solve (29) by conjugate gradient, or Jacobi, or Gauss-Seidel, or successive over-relaxation methods, with stopping rule (35).

Remark 9 If F is twice Fréchet differentiable, there are more options for solving equations (3) and (33): they can be solved by gradient-type methods, Newton-type methods, or a combination of these methods.

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